Lecture 14. Nonhomogeneous Equations and Undetermined Coefficients

Consider the general nonhomogeneous nth-order linear equation with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \quad (1)$$

A general solution of Eq.(1) has the form

$$\boxed{y = y_c + y_p}$$

where the complementary function $y_c(x)$ is a general solution of the associated homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

and $y_p(x)$ is a particular solution of Eq. (1).

Method of Undetermined Coefficients

Example 1 Find a general solution y of the given equation. ($f(x)$ is a polynomial.)

$$y'' + 4y = 3x^2 \quad (= f(x)) \quad ⊛$$

ANS: We have $y(x) = y_c + y_p$, where y_c is a genera solution of $y'' + 4y = 0$, y_p is a particular solution to ⊛.

- Find y_c. The char. eqn is $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

  Thus $y_c(x) = C_1 \cos 2x + C_2 \sin 2x$

- Find y_p. Guess $y_p = Ax^2 + Bx + C$ for some scalar A, B, C. then

  $y_p' = 2Ax + B$, $y_p'' = 2A$

  Plug them into ⊛

Then $y_p'' + 4y_p = 3x^2$

$$\Rightarrow 2A + 4(Ax^2 + Bx + C) = 3x^2$$

$$\Rightarrow 4Ax^2 + 4Bx + (2A + 4C) = 3x^2$$

By compairing the coefficients for x^2, x, & constants on both sides of the eqn.

$$\Rightarrow \begin{cases} 4A = 3 \\ 4B = 0 \\ 2A + 4C = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{3}{4} \\ B = 0 \\ C = -\frac{3}{8} \end{cases}$$

So $y_p = \frac{3}{4}x^2 - \frac{3}{8}$

Thus the general solution to the eqn is

$$y = y_c + y_p = C_1 \cos 2x + C_2 \sin 2x + \frac{3}{4}x^2 - \frac{3}{8}$$

Example 2 Find a particular solution y_p of the given equation. ($f(x)$ is an exponential fuction e^{rx}.)

$$y'' - 3y' - 4y = 3e^{2x}$$

ANS: We guess $y_p(x) = Ae^{2x}$. Then we need to figure out $A=?$ by plugging $y_p(x) = Ae^{2x}$ into the eqn. $y_p' = 2Ae^{2x}$, $y_p'' = 4Ae^{2x}$.

Then $y_p'' - 3y_p' - 4y_p = 3e^{2x}$

$\Rightarrow 4Ae^{2x} - 6Ae^{2x} - 4Ae^{2x} = 3e^{2x}$

$\Rightarrow -6A = 3 \Rightarrow A = -\frac{1}{2}$

Thus $y_p = -\frac{1}{2}e^{2x}$

Example 3 Find a particular solution y_p of the given equation. ($f(x)$ is $\cos kx$ or $\sin kx$.)

$$y'' - 3y' - 4y = 2\sin x$$

ANS: We guess $y_p = A\sin x + B\cos x$

$y_p' = A\cos x - B\sin x$

$y_p'' = -A\sin x - B\cos x$

Plug them into the eqn. $y_p'' - 3y_p' - 4y_p = 2\sin x$

$\Rightarrow -A\sin x - B\cos x - 3(A\cos x - B\sin x) - 4(A\sin x + B\cos x) = 2\sin x$

$\Rightarrow (-A + 3B - 4A)\sin x + (-B - 3A - 4B)\cos x = 2\sin x$

$\Rightarrow \begin{cases} -5A + 3B = 2 \\ -3A - 5B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{5}{17} \\ B = \frac{3}{13} \end{cases}$ $y_p = -\frac{5}{17}\sin x + \frac{3}{17}\cos x$

Example 4 Find a particular solution y_p of the given equation. ($f(x)$ is $e^{rx}\cos kx$ or $e^{rx}\sin kx$)

$$y'' - 3y' - 4y = -8e^x \cos 2x$$

ANS: We guess: $y_p(x) = Ae^x\cos 2x + Be^x \sin 2x$

$$y_p'(x) = A(e^x\cos 2x - 2e^x\sin 2x) + B(e^x \sin 2x + 2e^x\cos 2x)$$
$$= (A+2B)e^x\cos 2x + (-2A+B)e^x\sin 2x$$

$$y_p''(x) = (-3A+4B)e^x\cos 2x + (-4A-3B)e^x\sin 2x$$

Plug them into the eqn, we have

$$-8e^x\cos 2x = y_p'' - 3y_p' - 4y_p = (-3A+4B)e^x\cos 2x + (-4A-3B)e^x\sin 2x$$
$$-3[(A+2B)e^x\cos 2x + (-2A+B)e^x\sin 2x]$$
$$-4[Ae^x\cos 2x + Be^x\sin 2x]$$

By comparing the coefficients, we have

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{10}{13} \\ B = \frac{2}{13} \end{cases}$$ Thus $y_p = \frac{10}{13}e^x\cos 2x + \frac{2}{13}e^x\sin 2x$

Example 5 Find a particular solution y_p of the given equation. ($f(x)$ is a combination)

$$y'' - 3y' - 4y = \underbrace{3e^{2x}}_{f_1(x)} + \underbrace{2\sin x}_{f_2(x)} - \underbrace{8e^x\cos 2x}_{f_3(x)}$$

ANS: We can split the eqn into 3 eqns.

$$y'' - 3y' - 4y = 3e^{2x}$$

$$y'' - 3y' - 4y = 2\sin x$$

$$y'' - 3y' - 4y = -8e^x\cos 2x$$

Then by Examples 2-4, we have

$$y_p = \underbrace{-\frac{1}{2}e^{2x}}_{\text{Example 2}} \underbrace{- \frac{5}{17}\sin x + \frac{3}{17}\cos x}_{\text{Ex 3}} + \underbrace{\frac{10}{13}e^x\cos 2x + \frac{2}{13}e^x\sin 2x}_{\text{Ex 4}}$$

The Case of Duplication

Example 6 Find a particular solution of $y'' - 4y = 2e^{2x}$.

ANS: If we try ~~$y_p = Ae^{2x}$~~, then $y_p'' = 4Ae^{2x}$

Then $y_p'' - 4y_p = (4A-4A)e^{2x} = 0 \neq 2e^{2x}$

Why? The reason is the char eqn for the corresponding hom. eqn is $r^2 - 4 = 0 \Rightarrow r = \pm 2$. Thus $y = Ae^{2x}$ is a solution to the eqn $y'' - 4y = 0$

How to find a y_p in this case?

If we try $y_p = Axe^{2x}$ then

$y_p' = A(e^{2x} + 2xe^{2x})$, $y_p'' = A(2e^{2x} + 2e^{2x} + 4xe^{2x}) = 4Ae^{2x} + 4Axe^{2x}$

Then $y_p'' - 4y_p = 4Ae^{2x} + \cancel{4Axe^{2x}} - \cancel{4Axe^{2x}} = 2e^{2x}$ ($f(x)$)

$\Rightarrow 4A = 2 \Rightarrow A = \frac{1}{2}$

Then $y_p(x) = \frac{1}{2}xe^{2x}$ → case $s=1$

Polynomial of degree m Eg. $a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$

If the function $f(x)$ is of either form of $P_m(x)e^{rx}\cos kx$, $P_m(x)e^{rx}\sin kx$, we can assume

$$y_p(x) = x^s[(A_0 + A_1x + \cdots + A_mx^m)e^{rx}\cos kx + (B_0 + B_1x + \cdots + B_mx^m)e^{rx}\sin kx]$$

where s is the smallest nonnegative integer such that no term in y_p duplicates a term in the complementary function y_c.

Summary

We summarize the steps of finding the solution of an initial value problem consisting of a nonhomogeneous equation of the form

$$ay'' + by' + cy = f(x) \tag{2}$$

where a, b, c are constants, together with a given set of initial conditions:

1. Find the general solution of the corresponding homogeneous equation.
2. Make sure that function $f(x)$ in Eq. (2) belongs to the class of functions discussed above; that is, be sure it involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of such functions. If this is not the case, use the method of variation of parameters (discussed in the following part of this section).
3. If $f(x) = f_1(x) + \cdots + f_n(x)$, that is, if $f(x)$ is a sum of n terms, then form n subproblems, each of which contains only one of the terms $f_1(x), \ldots, f_n(x)$. The ith subproblem consists of the equation

   $$ay'' + by' + cy = f_i(x)$$

   where i runs from 1 to n. (see Example 5)
4. For the ith subproblem assume a particular solution $y_{ip}(x)$ consisting of the appropriate exponential function, sine, cosine, polynomial, or combination thereof. If there is any duplication in the assumed form of $y_{ip}(x)$ with the solutions of the homogeneous equation (found in step 1), then multiply $y_{ip}(x)$ by x, or (if necessary) by x^2, so as to remove the duplication. (See the table for general cases)
5. Find a particular solution $y_{ip}(t)$ for each of the subproblems. Then the sum $y_{1p}(t) + y_{2p}(t) + \cdots + y_{np}(t)$ is a particular solution of the full nonhomogeneous Eq (2).
6. Form the sum of the general solution of the homogeneous equation (step 1) and the particular solution of the nonhomogeneous equation (step 5). This is the general solution of the nonhomogeneous equation.
7. Use the initial conditions to determine the values of the arbitrary constants remaining in the general solution.

$f(x)$	y_p
$P_m = b_0 + b_2 x + \cdots + b_m x^m$	$x^s(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)$
$a \cos kx + b \sin kx$	$x^s(A \cos kx + B \sin kx)$
$e^{rx}(a \cos kx + b \sin kx)$	$x^s e^{rx}(A \cos kx + B \sin kx)$
$P_m(x) e^{rx}$	$x^s(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m) e^{rx}$
$P_m(x)(a \cos kx + b \sin kx)$	$x^s[(A_0 + A_1 x + x_2 x^2 + \cdots + A_m x^m) \cos kx +$ $(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m) \sin kx]$

Exercise 7 Set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.

(1) $\dfrac{d^2y}{dx^2} + 4y = x - \dfrac{x^2}{20}$

(2) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 8y = e^{2x}$

(3)$y'' + 4y' + 13y = 3\cos 2x$

(4) $y'' - 2y' - 15y = 3x\cos 2x$

Solutions.

(1). $y_p(x) = Ax^2 + Bx + C$

(2) $y_p(x) = Ae^{2x}$

(3) $y_p(x) = A\cos 2x + B\sin 2x$,

(4) $y_p(x) = (Ax + B)\cos 2x + (Cx + D)\sin 2x$

Exercise 8. Find y as a function of x if

$$y''' - 8y'' + 15y' = 24e^x,$$
$$y(0) = 20, \quad y'(0) = 20, \quad y''(0) = 17$$

Solution.

Step 1. Find the solution y_c for the corresponding homogeneous equation.

The homogeneous equation associated with the given ODE is:

$$y''' - 8y'' + 15y' = 0$$

The corresponding characteristic equation:

$$r^3 - 8r^2 + 15r = 0$$

The roots of the characteristic equation are $r = 0, r = 3$, and $r = 5$.

This gives us the complementary function to the homogeneous equation:

$$y_c(x) = C_1e^{0x} + C_2e^{3x} + C_3e^{5x} = C_1 + C_2e^{3x} + C_3e^{5x}$$

Step 2. Finding a particular solution.

Given the non-homogeneous term $24e^x$, we propose a particular solution of the form $y_p = Ae^x$. To determine A, we substitute y_p and its derivatives into the nonhomogeneous ODE and solve for A.

We have

$$Ae^x - 8Ae^x + 15Ae^x = 8Ae^x = 24e^x$$

Thus $A = 3$.

So $y_p = 3e^x$.

Step 3. Applying initial conditions

To determine the constants C_1, C_2, and C_3, we apply the initial conditions $y(0) = 20$, $y'(0) = 20$, and $y''(0) = 17$.

We have the general solution

$y(x) = y_c(x) + y_p(x) = C_1e^{0x} + C_2e^{3x} + C_3e^{5x} + 3e^x = C_1 + C_2e^{3x} + C_3e^{5x} + 3e^x$

As $y(0) = 20$, we have

$$C_1 + C_2 + C_3 + 3 = 20$$

As $y'(0) = 20$ and $y'(x) = 3C_2e^{3x} + 5C_3e^{5x} + 3e^x$, we have

$y'(0) = 3C_2 + 5C_3 + 3 = 20$

As $y''(0) = 17$, and and $y''(x) = 9C_2e^{3x} + 25C_3e^{5x} + 3e^x$, we have

$y'(0) = 9C_2 + 25C_3 + 3 = 17$.

Solve for $C_1,\ C_2,$ and C_3, we have

$C_1 = \frac{133}{15}, C_2 = \frac{71}{6}$, and $C_3 = -\frac{37}{10}$

Therefore, we have

$y(x) = \frac{133}{15} + \frac{71}{6}e^{3x} - \frac{37}{10}e^{5x} + 3e^x$

Exercise 9. Find a particular solution to

$$y'' + 6y' + 5y = -19te^{3t}.$$

Solution

Note the homogenous part of the ODE is

$y'' + 6y' + 5y = 0$ with characteristic equation $r^2 + 6r + 5 = 0$.

Thus the general solution is a linear combination of e^{-5t} and e^{-t} for the homogenous part.

So we assume $y_p(t) = a_1e^{3t} + a_2e^{3t}t$.

Then $y_p' = 3e^{3t}a_1 + e^{3t}a_2 + 3e^{3t}ta_2$ and

$y_p'' = \left(9e^{3t}a_1 + \left(6e^{3t} + 9e^{3t}t\right)a_2\right.$

Substitute the particular solution $y_p(t)$ into the differential equation:

$$\begin{aligned} & y_p'' + 6y_p' + 5y_p \\ =& 9a_1e^{3t} + a_2\left(6e^{3t} + 9e^{3t}t\right) + 6\left(3a_1e^{3t} + a_2e^{3t} + 3a_2e^{3t}t\right) + 5\left(a_1e^{3t} + a_2e^{3t}t\right) \\ =& -19e^{3t}t \end{aligned}$$

Simplify:

$$(32a_1 + 12a_2)e^{3t} + 32a_2e^{3t}t = -19e^{3t}t$$

Equate the coefficients of e^{3t} on both sides of the equation:

$$32a_1 + 12a_2 = 0$$

Equate the coefficients of $e^{3t}t$ on both sides of the equation:

$$32a_2 = -19$$

Solve the system:

$$a_1 = \frac{57}{256}$$
$$a_2 = -\frac{19}{32}$$

Substitute a_1 and a_2 into $y_p(t) = e^{3t}a_1 + e^{3t}ta_2$:

$$y_p(t) = \frac{57e^{3t}}{256} - \frac{19}{32}e^{3t}t$$

Exercise 10.

(1) Find a particular solution to the nonhomogeneous differential equation

$$y'' + 4y' + 5y = -10x + 5e^{-x}.$$

(2) Find the most general solution to the associated homogeneous differential equation.

(3) Find the most general solution to the original nonhomogeneous differential equation.

Solution.

(1)

Determine the particular solution to $\frac{d^2y(x)}{dx^2} + 4\frac{dy(x)}{dx} + 5y(x) = 5e^{-x} - 10x$ by the method of undetermined coefficients:

The particular solution will be the sum of the particular solutions to

$$y'' + 4y' + 5y = -10x \text{ and } y'' + 4y' + 5y = 5e^{-x}.$$

The particular solution to $y'' + 4y' + 5y = -10x$ is of the form:

$$y_{p_1}(x) = a_1 + a_2x$$

The particular solution to $y'' + 4y' + 5y = 5e^{-x}$ is of the form:

$$y_{p_2}(x) = a_3e^{-x}$$

Apply the usual step (I will skip this part), we have

$$a_1 = \frac{8}{5}$$
$$a_2 = -2$$
$$a_3 = \frac{5}{2}$$

Thus a particular solution is

$y_p(x) = \dfrac{5e^{-x}}{2} - 2x + \frac{8}{5}$

(2) The homogenous equation is

$$y'' + 4y' + 5y = 0$$

The characteristic equation is

$$r^2 + 4r + 5 = 0$$

Then $r = -2 + i$ or $r = -2 - i$

So we have

$y_c = c_1 e^{-2x} \cos(x) + c_2 e^{-2x} \sin(x)$

(3)

By (1) and (2), we have

$$y(x) = y_c(x) + y_p(x) = c_1 e^{-2x} \cos(x) + c_2 e^{-2x} \sin(x) + \frac{5e^{-x}}{2} - 2x + \frac{8}{5}$$